KUWAIT UNIVERSITY
Department of Mathematics & Computer Science

Math 102 | | 22 Jan 2009
Calculus B | Final Exam | Two hours

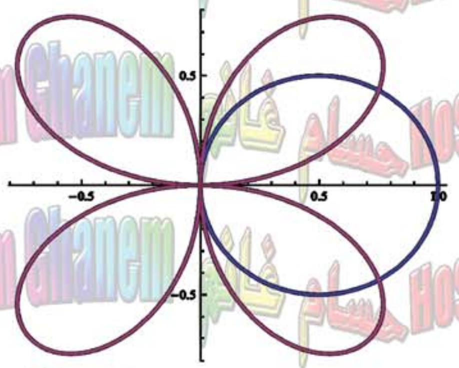
Calculators and mobile phones are not allowed in the exam.

1. Let f be a one-to-one differentiable function such that $f(2) = 3$, $f'(2) = 5$, $f'(3) = 4$. If $g = f^{-1}$, what is $g'(3)$? [2 pts]

2. Show that if f is one-to-one, then $g(x) = e^{f(x)}$ is also one-to-one. [2 pts]

3. Show that if $\cos(\arctan a) = \tan(\arccos a)$, then $2a^2 = \sqrt{5} - 1$. [3 pts]

4. The figure below shows the graphs of the polar equations $r = \cos\theta$ and $r = \sin 2\theta$. Find the area of the region that lies inside both curves. [4 pts]



5. A curve C has parametrization $x = 2t^3 - 6t$; $y = 2t^3 + 3t^2$, where $-1 \le t \le 1$. Find the coordinates of the points on C at which the tangent line has slope $1/3$. [3 pts]

6. Find the arc length of the curve C whose parametric equations are [3 pts]

$$x = \ln\sqrt{1+t^2}, \qquad y = \tan^{-1} t, \qquad t \in [0,1]$$

7. Evaluate the following integrals: [16 pts]

$$\text{(a)} \int \frac{\sin x}{1 + \sin x - \cos x}\,dx \qquad \text{(b)} \int \frac{1}{x^2}\sin^{-1} x\,dx$$

$$\text{(c)} \int \frac{4}{\sin^2 x + (1+\cos x)^2}\,dx \qquad \text{(d)} \int \frac{\sqrt[4]{x+9}}{1+\sqrt{x+9}}\,dx$$

8. Does the following integral converge or diverge? If it converges, find its value. [4 pts]

$$\int_{-\pi/2}^{\pi/2} \frac{\sin 2x}{\sqrt[3]{\sin^4 x}}\,dx$$

9. Answer as true (T) or false (F) and justify your answer. [3 pts]

   (a) If f is continuous, then f is one-to-one.

   (b) If $(\ln a)^2 = (\ln b)^2$, then $a = b$.

Solutions:

1. $g(3) = 2$ since $f(2) = 3$. Hence, $g'(3) = 1/f'(g(3)) = 1/f'(2) = 1/5$.

2. Using the fact that f and e^x are one-to-one functions, if $x_1 \neq x_2$ then $f(x_1) \neq f(x_2)$ and hence $e^{f(x_1)} \neq e^{f(x_2)}$. Therefore, by definition, e^f is one-to-one.

3. $\theta_1 = \arctan a \quad \to \quad \tan\theta_1 = a \quad \to \quad \cos\theta_1 = 1/\sqrt{1+a^2}$ and $\theta_2 = \arccos a \quad \to \quad \cos\theta_2 = a \quad \to \quad \tan\theta_2 = \sqrt{1-a^2}/a$. Equate the two to get $a^2 = 1 - a^4$ or $a^4 + a^2 - 1 = 0$, then solve for a^2.

4. The intersection points we obtain from $\cos\theta\,(1 - 2\sin\theta) = 0$. By symmetry, we consider the first quadrant, where the curves intersect at $\theta = \pi/2$ and $\theta = \pi/6$. The area inside both curves becomes

$$A = 2(A_1 + A_2) = 2\left(\frac{1}{2}\int_0^{\pi/6} \sin^2 2\,\theta\, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} \cos^2\theta\, d\theta\right) \quad \to$$

$$A = \frac{1}{2}\int_0^{\pi/6} (1 - \cos 4\theta)\, d\theta + \frac{1}{2}\int_{\pi/6}^{\pi/2} (1 + \cos 2\theta)\, d\theta = \cdots = \frac{\pi}{4} - \frac{3}{8}\sin\frac{\pi}{3} = \frac{\pi}{4} - \frac{3\sqrt{3}}{16}$$

5. $x = 2t^3 - 6t \ \to\ \dot{x} = 6t^2 - 6$; $y = 2t^3 + 3t^2 \ \to\ \dot{y} = 6t^2 + 6t$. The slope is given by

$$m(t) = \frac{dy}{dx} = \frac{\dot{y}}{\dot{x}} = \frac{t^2 + t}{t^2 - 1} = \frac{t\,(t+1)}{(t-1)\,(t+1)} = \frac{t}{t-1}, \qquad t \neq -1$$

From $m(t) = 1/3$ we get $t = -1/2$ and the point where the slope is $1/3$ has coordinates $(11/4, 1/2)$.

6. $x = (1/2)\,\ln(1+t^2) \ \to\ \dot{x} = t/(1+t^2)$; $y = \tan^{-1} t \ \to\ \dot{y} = 1/(1+t^2)$. The arc length:

$$L = \int_0^1 \frac{\sqrt{1+t^2}}{1+t^2}\, dt = \int_0^1 \frac{1}{\sqrt{1+t^2}}\, dt = \int_0^{\pi/4} \sec\theta\, d\theta = \ln(1+\sqrt{2})$$

using the trigonometric substitution $t = \tan\theta$.

7. (a) Think like Weierstrass and put $u = \tan(x/2)$:

$$\int \frac{\sin x}{1 + \sin x - \cos x}\, dx = \int \frac{2}{(u^2+1)\,(u+1)}\, du = \int \left[\frac{1}{u+1} + \frac{1-u}{u^2+1}\right] du$$

$$= \ln|u+1| - \frac{1}{2}\ln(u^2+1) + \tan^{-1} u + C$$

$$= \ln|\tan(x/2) + 1| - \frac{1}{2}\ln\left(\tan^2(x/2) + 1\right) + \tan^{-1}(\tan(x/2)) + C$$

(b) Integrate by parts with $u = \sin^{-1} x$ and $dv = dx/x^2$:

$$\int \frac{\sin^{-1} x}{x^2}\, dx = -\frac{\sin^{-1} x}{x} + \int \frac{1}{x\sqrt{1-x^2}}\, dx$$

Put $x = \sin\theta$:

$$\int \frac{1}{x\sqrt{1-x^2}}\, dx = \int \frac{1}{\sin\theta\,\cos\theta}\,\cos\theta\, d\theta = \int \csc\theta\, d\theta = \ln|\csc\theta - \cot\theta| + C$$

$$= \ln|1-\sqrt{1-x^2}| - \ln|x| + C \quad \rightarrow$$

$$\int \frac{\sin^{-1} x}{x^2}\, dx = -\frac{\sin^{-1} x}{x} + \ln|1-\sqrt{1-x^2}| - \ln|x| + C$$

(c) $\sin^2 x + (1+\cos x)^2 = 1 - \cos^2 x + 1 + 2\cos x + \cos^2 x = 2 + 2\cos x$ so that

$$\int \frac{4}{\sin^2 x + (1+\cos x)^2}\, dx = \int \frac{2}{1+\cos x}\, dx = 2\int \frac{1-\cos x}{\sin^2 x}\, dx$$

$$= 2\int (\csc^2 x - \csc x\, \cot x)\, dx = 2\,(\csc x - \cot x) + C$$

(d) Let $u^4 = x + 9$ then $dx = 4u^3$ and the integral becomes

$$\int \frac{\sqrt[4]{x+9}}{1+\sqrt{x+9}}\, dx = 4\int \frac{u^4}{1+u^2}\, du = 4\int \left(u^2 - 1 + \frac{1}{1+u^2}\right) du$$

$$= 4\left(\frac{u^3}{3} - u + \tan^{-1} u\right) + C = 4\left(\frac{(\sqrt[4]{x+9})^3}{3} - \sqrt[4]{x+9} + \tan^{-1}(\sqrt[4]{x+9})\right) + C$$

8. Do the integral first

$$\int \frac{\sin 2x}{\sqrt[3]{\sin^4 x}}\, dx = \int 2\, \frac{\sin x}{(\sin x)^{4/3}}\, \cos x\, dx = 2\int (\sin x)^{-1/3} \cos x\, dx = 3\,(\sin x)^{2/3}$$

The definite integral becomes

$$\int_{-\pi/2}^{\pi/2} \cdots = \int_{-\pi/2}^{0} \cdots + \int_{0}^{\pi/2} \cdots = \lim_{s\to 0^-} \int_{-\pi/2}^{s} \cdots + \lim_{t\to 0^+} \int_{t}^{\pi/2} \cdots = -3 + 3 = 0$$

9. (a) False: $f(x) = \sin x$ is continuous on $\mathbb{R}$ but not one-to-one.

(b) False: a may equal to either b or $1/b$.